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Delay, feedback and quenching in financial markets

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Abstract. An asset whose price exhibits geometric Brownian motion is analysed. The basic Brownian motion model is modified to account for the effects of market delay and investor feedback. A Langevin equation model is appropriate. When the feedback coupling is sufficiently strong, the market dynamics switches from a slow random walk behaviour to a rapid unstable behaviour with a fast time scale characteristic of the market delay. The unstable runaway behaviour is subsequently quenched by investors deserting a collapsing market or saturating a booming one. This quenching effect is sufficient to ensure long term bounding of the asset price. A form of market sabotage is demonstrated in which investors can push the market from a stable to an unstable regime.

PACS. 02.50.Ey Stochastic processes – 89.90.+n Other areas of general interest to physicists

1 Introduction

The year 2000 marks the centenary of the publication of Bachelier's classic treatise on price dynamics in financial markets [1,2]. The basic premise of this work is that asset prices undergo a geometric Brownian motion [3]. Thus for an asset price P and time t, one assumes

$$\frac{\mathrm{d}\log P}{\mathrm{d}t} = \nu + \sigma\Xi(t), \qquad (1)$$

where log denotes the *natural* logarithm, the constants ν and σ are respectively mean growth rate and market volatility, and $\Xi(t)$ is a random white noise process. A consequence of equation (1) is that, if $\log P$ has a known initial value $\log P_0$, thereafter the variance of $\log P$ scales like $\sigma^2 t$, and thus σ^2 represents a price diffusivity.

This geometric Brownian motion model has met with some spectacular successes. Many modern theories of option pricing [4] are based on a derivation that implicitly assumes an underlying geometric Brownian motion. Nonetheless it is still valid to ask whether a more realistic model of asset price dynamics might be available, which describes markets better than equation (1). When seeking new, more realistic financial models, researchers have often turned to the physical world for inspiration, since the study of Brownian phenomena in physics is already quite mature. This practice has led to the emerging discipline of "econophysics" [5–8].

One of the key elements of real markets which equation (1) fails to describe is the feedback between investor activity determining supply and demand and hence affecting asset price. Thus while equation (1) might be appropriate to describe a situation during which investors are acting more or less randomly or independently of one another, it would not describe those periods when investors were acting in concert or collectively. This sort of collective behaviour would be the basis of a speculative boom, a stock market crash or a currency crisis.

A recently formulated model [9,10] has attempted to address these feedback related issues. It divides investors into two groups: fundamentalists, who perceive that a each asset has a definite or fundamental value regardless of its actual market price, and chartists, who merely follow market trends. The fundamentalists tend to stabilize markets, by restoring the actual market price to the fundamental value, while the chartists destabilize them, through their collective behaviour or herd mentality. Following the practice in statistical physics, transition probabilities were assigned between fundamentalist and chartist groups, with the actual probabilities depending on market conditions. The model discovered intermittent noisy bursts, during which the actual price fluctuated over a wide amplitude. This was despite the fact that incoming news about the asset, which influences the fundamental value, exhibited no such bursts. Instead the bursts were identified with periods of unusually high population in the chartist group. The bursts were short lived, because the fundamentalists tended to stabilize the market whenever the actual asset price deviated too far from the fundamental value. Feedback-induced bursting also caused an anomalously high statistical weight to be assigned to the tail of the distribution of short term price changes, a contrast from the Gaussian distributions produced when feedback is absent.

Various other financial models exist in the physics literature which incorporate elements of collective behaviour or feedback [11–20]. These models exhibit similar bursts and again produce anomalous non-Gaussian

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shapes for the price change distribution. There is some disagreement in the literature as to precisely which weight should be assigned to these non-Gaussian tails. Some suggestions are the Lévy distribution [21,22], various power laws [9,23,24], stretched exponentials [25], and simple exponential tails [26–29]. A very good candidate is the so called truncated Lévy distribution [26–29], which essentially gives a power law in the near tail and a simple exponential in the far tail.

Another aspect of real markets that one might seek to describe is delay. The market itself and the investors within it take a small but finite time to respond to incoming news affecting an asset. With the advent of internet technology and computer trading these response times have shrunk dramatically, say to the order of minutes. Certainly this time scale is small compared with the diffusive time scale associated with a pure geometric Brownian motion, which may be on the order of years. Indeed a speculative boom or a market crash may be said to be associated with a shift from dynamical behaviour on the slow diffusive time scale to dynamics controlled by the very much smaller delay time.

These effects have been modelled by Bouchaud and Cont [29,30] again applying ideas from physics. They have used as their basis a Langevin model, employed in physics to describe Brownian particles with inertia [31–37]. Such particles possess two governing time scales, an inertial relaxation time and a diffusive time, with the former often much smaller than the latter. The particle velocity, rather than depending on the instantaneous Brownian force, as would be the case for a pure inertialess Brownian motion, depends instead on all the random forces over roughly one inertial time scale in the recent past. The particle position however depends on the entire forcing history, except for forcing in the recent past, which has not yet realized its full influence. Thus an inertial particle executes a random walk very similar to that of an inertialess particle, but it is less jagged, being smoothed on the order of an inertial relaxation time [36,37], and it also lags behind the corresponding inertialess particle by a comparable amount.

The financial Langevin model [29,30] includes feedback effects in addition to delay. In the governing equations, positive feedback from chartist-like behaviour affects the market relaxation rate term, which is analogous to friction in the Langevin equation. Positive feedback reduces relaxation rate, giving a heightened sensitivity to random forcing, along with more persistent random effects. If the feedback is strong enough, the net "friction" reaches a critical point where it switches from positive to negative. In the financial context, this means that the act of buying a block of shares, forces the price up, leading to a further share purchase even larger than the original, and so on. The time between each subsequent round of buying is on the order of the market delay time, so the asset price runs away on this time scale. The dynamics are no longer governed by the diffusive time scale of the geometric Brownian motion, but instead by a much smaller delay time.

There were a number of other effects added to the financial Langevin model. A fundamentalist type term was included representing the true value of an asset, rather than its current market price. Inherently unstable markets were found to support speculative bubbles during which growth of the market price bore little relation to true value. These inevitably burst when the price-value discrepancy became insupportable. Risk aversion was also present, *i.e.* investors choose non-volatile in preference to volatile markets, and in addition they buy rising assets cautiously, but sell falling ones immediately. This implied that even in a nominally stable market (*i.e.* one with positive "friction" for small fluctuations), crashes could still occur, being induced by rare events (large negative price fluctuations) frightening the investors.

The purpose of the present paper is to investigate a particularly simple Langevin model for an asset price, involving delay and positive feedback, along with a mechanism for stopping unstable markets from running away indefinitely. The aims are both to study the characteristics of individual trajectories in stable and unstable regimes, and to analyse the statistics of log price and its rate of change. Stability criteria will be established, and also a procedure by which chartists can sabotage markets by modulating their own behaviour. No explicit fundamentalist term acting to restore prices to some true value will be included here. Instead we rely on instabilities being controlled *via* a weaker mechanism consisting of chartists deactivating once they have either abandoned or saturated a market. For convenience we shall refer to this control mechanism as quenching. It is found that quenching on its own is sufficient to ensure well-behaved markets in the long term.

The structure of the rest of the paper is as follows. In the next section we introduce and derive the Langevin model to be considered. Following that we consider a number of theoretical results that can be derived using the model. The paper then turns to numerical issues: first with a discussion of suitable numerical algorithms, and then with results. Finally concluding remarks are made, and possible extensions to the model are considered.

2 Model with delay, feedback and quenching

In this section we formulate the model to be used in this paper. As stated above our aim is to understand delay, feedback and quenching, and so we have chosen the barest possible model capable of describing those effects. As we are interested in possible market instabilities, and it is chartists rather than fundamentalists who are the root cause of these, no fundamentalist type terms will be included in the model. Likewise, in the interests of simplicity, we consider a market that is static on average. There is no mean growth term to make shares increasingly attractive over time, nor any interest rate, which discriminates against share buying in favour of "riskless" investments. The rationale for ignoring these terms in the present instance is that we want to focus attention on the rapid time scale events (stock market booms and crashes) that have dynamics on the order of one delay time. Mean growth and interest rates are longer time scale effects, which may influence how often the rapid catastrophic events occur, but play no role in the dynamics of the rapid events themselves.

The model has two essential ingredients. First it must describe how an asset price is affected by chartists, and second how chartists respond to asset price.

We shall use the symbol S to denote the proportion of available chartist capital currently invested in the asset under question. For simplicity we shall refer to S as the saturation, representing as it does the extent to which chartists have become saturated.

If the chartists take the initiative in share trading, price may be assumed to be forced by the rate $\mathrm{d}S/\mathrm{d}t$ at which they acquire or dispose of their portfolios. There is a proportionality factor here, which we denote by the symbol M, which determines the market sensitivity to chartists. In addition to this chartist feedback term, we assume that price is still affected by incoming random news about the asset, which will continue to be modelled by a Brownian white noise.

Both the chartist feedback and the random effects are supposed not to affect the market immediately, but only after some appropriate delay. Throughout the paper we assume that time is made dimensionless based on the price diffusivity. The dimensionless delay time, which we denote Δt, will therefore be much smaller than unity.

Thus in place of the geometric Brownian motion described by equation (1), we have the dimensionless equation for evolution of log price

$$\Delta t \frac{\mathrm{d}^2 \log P}{\mathrm{d}t^2} + \frac{\mathrm{d} \log P}{\mathrm{d}t} = \Xi(t) + M \frac{\mathrm{d}S}{\mathrm{d}t} \cdot \qquad (2)$$

Here Ξ is white noise satisfying $\langle \Xi(t) \rangle = 0$ for all t, and

$$\langle \Xi(t)\Xi(t') \rangle = 2\delta(t - t'), \qquad (3)$$

for times t and t' with $\delta(t - t')$ denoting the Dirac delta function. This white noise can be realized in a computer simulation as follows. First choose a time step δt much smaller than any other time scale in the problem, in this case much smaller than the delay Δt. Then for each time step assign a random number ξ with zero mean and variance two. Finally define

$$\Xi = \frac{\xi}{\delta t^{1/2}} \cdot \qquad (4)$$

The second part of the model must describe how the chartists respond to price changes. We assume that their rate of acquiring or disposing of assets simply follows the rate of change of log price. A constant of proportionality R measures how responsive the chartists are to market conditions. We have the dimensionless equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} = R \frac{\mathrm{d} \log P}{\mathrm{d}t} \cdot \qquad (5)$$

Note that, in the interests of simplicity, no investor delay has been incorporated in equation (5). Such an investor, who can respond to changing conditions faster than the market as a whole, could clearly devise investment strategies superior to equation (5), and thereby maximize earnings. However the aim in this paper is not to investigate strategies, but rather to model the sort of chartist collective behaviour that can lead to market booms and crashes. At this level of modelling, it is only considered important that there be a delay somewhere in the model, and that this delay time be much smaller than the time scale of the pure geometric Brownian motion.

There are however important budgetary constraints on the model at $S = 0$ (lower constraint boundary) and $S = 1$ (upper constraint boundary). At $S = 0$ chartists no longer have assets to sell to create excess supply, while at $S = 1$ they no longer have funds to commit to create excess demand. For simplicity we assume that these boundaries at $S = 0$ and $S = 1$ are hard: speculators are prepared to withdraw entirely from the market, but they are not prepared to sell assets they do not yet possess; likewise they will risk their entire capital, but will not borrow money to continue speculating beyond this point. Thus at $S = 0$ or $S = 1$ precisely, equation (5) ceases to be valid, and the feedback term in equation (2) switches off. In the absence of feedback, runaway prices tend to decelerate, and this is the process we refer to as quenching. Feedback will only switch back on again after a complete deceleration when $\mathrm{d} \log P/\mathrm{d}t$ changes sign. This reactivates the chartists, and moves the saturation S away from the constraint boundaries. It is conceivable while attempting to leave a boundary, S may exhibit a number of false starts, beginning to move away from the constraint, but failing to reach its normal speed, and then crashing back into the boundary. We shall use the term dither to refer to this phase of the dynamics, and analyse it more closely in a later section.

To summarize, a model has been derived which contains the desired ingredients of market delay, positive feedback and a mechanism for quenching runaway markets. The derivation has ignored investor delay, mean market growth, interest rates and the influence of fundamentalists. These will be considered briefly in Section 6.

2.1 Derivation of a Langevin equation

Away from constraint boundaries, we can substitute equation (5) into equation (2) to obtain a Langevin equation for $\log P$

$$\Delta t \frac{\mathrm{d}^2 \log P}{\mathrm{d}t^2} + \frac{\mathrm{d} \log P}{\mathrm{d}t} = \Xi(t) + MR \frac{\mathrm{d} \log P}{\mathrm{d}t} \cdot \qquad (6)$$

The Langevin equation derived here is very similar to that ultimately derived by Bouchaud and Cont [30]. The underlying models are slightly different. There is no delay term nor randomness directly affecting price in the Bouchaud and Cont model, only a supply-demand type feedback. To compensate however those authors allow delay and randomness to regulate investor behaviour, which is not the situation here.

3 Theoretical results

Langevin equation (6) is linear, and hence exactly solvable given a sequence of random forces. This fact along with the known statistics of the random forces can be used to derive a set of useful statistical results for $\log P$ and $\mathrm{d}\log P/\mathrm{d}t$.

The main observation about equation (6) concerns its stability. The $\mathrm{d}\log P/\mathrm{d}t$ term on the left hand side describes the relaxation of the market and in physical terms plays the role of friction. The $MR\,\mathrm{d}\log P/\mathrm{d}t$ term on the right hand side is the positive feedback term, and has the effect of reducing the friction. A critical point is surpassed when the product MR exceeds unity. In that case rather than the effects of random forces gradually dying away, they become magnified over time. The market is then unstable. Random effects in the early history of the market determine its subsequent behaviour, and cannot be overcome by more recent forcing. The mechanism is as described earlier, namely the act of buying a block of shares, induces further share buying even greater than the original purchase, making the market run away extremely rapidly.

Suppose that the initial value of $\log P$ is $\log P_0$, and for simplicity that the initial value of $\mathrm{d}\log P/\mathrm{d}t$ vanishes. Well known results from statistical physics [31–33,36,37] for Langevin equation (6) can be used to determine the standard deviation $(\log P - \log P_0)_{\mathrm{rms}}$, the root mean square price velocity $(\mathrm{d}\log P/\mathrm{d}t)_{\mathrm{rms}}$, and the correlation coefficient between log price and velocity r_{corr}. Although results are available for arbitrary t values, most interest will focus on the regime $t \gg \Delta t$ so we give the asymptotic results for that case first. Quite different behaviour is found in the stable $MR < 1$ and unstable cases $MR > 1$ and we treat these separately.

3.1 The stable regime

In the stable case, we define a parameter

$$\Lambda_{\mathrm{s}} = 1 - MR, \tag{7}$$

and we find

$$(\log P - \log P_0)_{\mathrm{rms}} = \left(\frac{2t}{\Lambda_{\mathrm{s}}^2}\right)^{1/2}, \tag{8}$$

$$\left(\frac{\mathrm{d}\log P}{\mathrm{d}t}\right)_{\mathrm{rms}} = (\Lambda_{\mathrm{s}}\Delta t)^{-1/2}, \tag{9}$$

$$r_{\mathrm{corr}} = \left(\frac{\Delta t}{2\Lambda_{\mathrm{s}} t}\right)^{1/2}. \tag{10}$$

Note that equation (8) describes a diffusive motion, as one would obtain with a pure geometric Brownian motion. However the feedback has made the market more volatile: the price diffusivity has increased from unity to $\Lambda_{\mathrm{s}}^{-2}$. A small correction $-3\Delta t\,\Lambda_{\mathrm{s}}^{-3}$ can also be included on the right hand side of equation (8), which indicates that a market with delay lags slightly behind true Brownian motion.

Equation (9) indicates that price velocity is large, scaling as $\Delta t^{-1/2}$. It is noteworthy that this result is mathematically equivalent to the equipartition of energy in statistical mechanics: particles with less inertia (less delay), travel faster. Note moreover that for a pure geometric Brownian motion given by equation (1) with random force Ξ satisfying equation (4), the price velocity is larger still. It scales as $\delta t^{-1/2}$, where δt is assumed smaller than all other time scales in the problem. Thus equation (9) demonstrates the smoothing effect of the delay time Δt on the random motion.

Equation (10) indicates a small and decaying correlation between $\log P - \log P_0$ and $\mathrm{d}\log P/\mathrm{d}t$, so that knowing the current price is no help in determining whether price is actually increasing or decreasing. This happens because $\log P$ depends on the entire past forcing history, but $\mathrm{d}\log P/\mathrm{d}t$ only upon the recent past.

3.2 The unstable regime

In the unstable case we define

$$\Lambda_{\mathrm{u}} = MR - 1, \tag{11}$$

and then we find, still with $t \gg \Delta t$,

$$(\log P - \log P_0)_{\mathrm{rms}} = \left(\frac{\Delta t}{\Lambda_{\mathrm{u}}^3}\right)^{1/2} \exp\frac{\Lambda_{\mathrm{u}} t}{\Delta t}, \tag{12}$$

$$\left(\frac{\mathrm{d}\log P}{\mathrm{d}t}\right)_{\mathrm{rms}} = (\Lambda_{\mathrm{u}}\Delta t)^{-1/2} \exp\frac{\Lambda_{\mathrm{u}} t}{\Delta t}, \tag{13}$$

$$r_{\mathrm{corr}} = 1. \tag{14}$$

Runaway exponential growth can be readily seen for both $(\log P - \log P_0)_{\mathrm{rms}}$ and $(\mathrm{d}\log P/\mathrm{d}t)_{\mathrm{rms}}$. The exponential rate constant equals $\Lambda_{\mathrm{u}}/\Delta t$. A high correlation between $\log P - \log P_0$ and $\mathrm{d}\log P/\mathrm{d}t$ now exists, because early forcing alone determines the direction in which the market evolves. Later forcing cannot overcome this.

We emphasize that this behaviour is an unconstrained result based on linear equation (6). In reality the fact that S is constrained between 0 and 1, will rapidly limit this exponential growth.

3.3 The marginal stability case

In the marginal stability case $MR \to 1$ the following results apply

$$(\log P - \log P_0)_{\mathrm{rms}} = \left(\frac{2t^3}{3\Delta t^2}\right)^{1/2}, \tag{15}$$

$$\left(\frac{\mathrm{d}\log P}{\mathrm{d}t}\right)_{\mathrm{rms}} = \left(\frac{2t}{\Delta t^2}\right)^{1/2}, \tag{16}$$

$$r_{\mathrm{corr}} = \frac{\sqrt{3}}{2}. \tag{17}$$

In addition to $MR \to 1$, these results also apply to general values of Λ_{s} or Λ_{u} in the regime $t \ll \Delta t$, before the relaxation and feedback terms have had much opportunity to act. It can be seen from equations (15, 16) how delay makes the market much slower starting than a true Brownian motion. In equation (17) one sees that the correlation coefficient begins with an intermediate value, and it must either decay to zero (stable) or grow to unity (unstable) from there. In a numerical implementation, a few Brownian time steps are actually required before equations (15–17) are realized, but these first few steps may be neglected given $\delta t \ll \Delta t$.

3.4 Treatment of constraints

The theoretical results we have considered all pertain to the linear Langevin equation. Nonlinearities however arise due to the presence of the constraints at $S = 0$ and $S = 1$. A full treatment of the nonlinearities requires a numerical solution, discussed in Sections 4, 5. However a number of heuristic analytical results can be derived before proceeding with numerics.

First note that, although the Langevin equation only depends on M and R as a product MR, the separate values are important in determining the time at which the system encounters a constraint boundary. We note that to hit one constraint boundary starting from the other, S must change by unity, and hence by equation (5), $\log P$ must change by R^{-1}. A rough estimate of the time taken to cross from one boundary to the other can be obtained by equating $(\log P - \log P_0)_{\mathrm{rms}}$ to R^{-1}. If we use the symbol t_{c} to denote this crossing time, we find

$$t_{\mathrm{c}} \approx \tfrac{1}{2}\frac{(1-MR)^2}{R^2}, \tag{18}$$

in the stable regime, and

$$t_{\mathrm{c}} \approx \frac{\Delta t}{(MR-1)} \log \frac{(MR-1)^{3/2}}{R\,\Delta t^{1/2}} \approx O\left(\frac{\Delta t}{(MR-1)}\right), \tag{19}$$

when the system is unstable. In equation (19) it is expected that the logarithmic term remains near unity.

In the unstable system, once the crossing is underway, the system should remain unidirectional and hence

$$\left.\frac{\mathrm{d}\log P}{\mathrm{d}t}\right|^{\text{crossing}} = O\left((Rt_{\mathrm{c}})^{-1}\right) = O\left(\frac{(MR-1)}{R\Delta t}\right). \tag{20}$$

As the unstable trajectory is by and large determined once an initial fluctuation has moved it off the boundary, we could attempt to model this trajectory by

$$\log P - \log P_0 = \pm \log P_{\mathrm{i}} \exp \frac{\Lambda_{\mathrm{u}} t}{\Delta t}, \tag{21}$$

where $\log P_{\mathrm{i}}$ here represents the initiation fluctuation that starts the price moving.

Clearly in equation (21) the log price starts off moving slowly, but increases in speed as it approaches the opposite boundary. If the probability distribution of $\log P$ values scales inversely as the local speed, we can deduce a probability density $\mathcal{P}$ function for a single traverse

$$\mathcal{P} \propto (\log P - \log P_0)^{-1}, \tag{22}$$

in the domain $\log P > \log P_0 + \log P_{\mathrm{i}}$. As the changes in S follow those in $\log P$ and as the system is symmetric about the lower and upper boundaries, we guess a probability density averaged over many traverses

$$\mathcal{P} \propto S^{-1} + (1-S)^{-1}. \tag{23}$$

This is quite distinct from the stable case, where price velocity is uncorrelated with price itself. In that case a rather flatter probability distribution is expected.

If we identify $\log P - \log P_0$ from equation (21) with the unstable $(\log P - \log P_0)_{\mathrm{rms}}$ from equation (12) then we deduce

$$\log P_{\mathrm{i}} = \left(\frac{\Delta t}{\Lambda_{\mathrm{u}}^3}\right)^{1/2}. \tag{24}$$

If Λ_{u} is increased, traverses can be initiated by increasingly small fluctuations, and so the time spent dithering waiting for a suitable fluctuation should be less. A bound on the root mean square velocity during the dither phase can be obtained by differentiating equation (21) with respect to time, and evaluating at $t = 0$ just as the dither is ending and the crossing is about to begin. This produces a result scaling as $\Delta t^{-1/2}$, which is considerably less than the $O(\Delta t^{-1})$ velocity anticipated by equation (20) during a crossing.

4 Numerical algorithm

The estimates presented in the previous section have either ignored the constraints altogether, or treated them in a heuristic fashion. In order to take proper account of constraints a numerical solution is required. Bearing in mind the pitfalls associated with simulating constrained stochastic differential equations [35], great care has been taken to produce a numerical algorithm suitable for the particular equations that we consider in this paper. The principal difficulty with formulating a numerical algorithm of high order accuracy can be expressed as follows. In order to update log price accurately, one needs to know the price velocity and price acceleration. However owing to the on-off nature of the chartist feedback term, and the fact that the exact switching point is determined by price, in order to obtain the price acceleration, one actually needs to know the price in advance.

We have therefore adopted an algorithm at a level of accuracy such that only Brownian accelerations, not relaxation nor chartist feedback terms, are required in the update of $\log P$. An explicit numerical method is then feasible. We discuss this in three parts, the update of $\log P$, the update of S and the update of $\mathrm{d}\log P/\mathrm{d}t$. Those readers uninterested in the details of the algorithm, but only in the results, are advised to skip to Section 5.

4.1 Update of log P

The update of $\log P$ is the most straightforward part of the algorithm. Although we are ultimately only interested in values of time separated by intervals δt, it is convenient for what follows to define a continuous variable t' ranging between 0 and δt. We shall use the symbol $\delta \log P(t')$ to denote the change in $\log P$ between 0 and t', and reserve the symbol $\delta \log P$ with no argument to denote the change across the full step.

If $\mathrm{d}\log P/\mathrm{d}t|_0$ denotes the price velocity at the beginning of the step, the algorithm assumes

$$\delta \log P(t') = \left.\frac{\mathrm{d}\log P}{\mathrm{d}t}\right|_0 t' + \frac{\xi}{\delta t^{1/2} \Delta t}\,\tfrac{1}{2} t'^2, \qquad (25)$$

and hence

$$\delta \log P = \left.\frac{\mathrm{d}\log P}{\mathrm{d}t}\right|_0 \delta t + \tfrac{1}{2}\frac{\xi\,\delta t^{3/2}}{\Delta t}\cdot \qquad (26)$$

The rationale for retaining the Brownian price acceleration term, but discarding the relaxation and feedback terms, is the large magnitude of the Brownian force which scales as $\delta t^{-1/2}$. Remembering that $\mathrm{d}\log P/\mathrm{d}t$, at least for a stable market, is typically $O(\Delta t^{-1/2})$, the discarded terms would affect $\delta \log P$ by $O(\delta t^2 \Delta t^{-3/2})$, which is the per step error of the scheme.

4.2 Update of S

The update of the saturation S proceeds by first doing an unconstrained step, then checking whether the constraints on S are violated at any point in the step, and finally doing the actual update.

We define the unconstrained change $\delta S^{\mathrm{uncon}}(t')$ at an arbitrary time t' during the step to be

$$\begin{aligned}\delta S^{\mathrm{uncon}}(t') &= R\,\delta \log P(t') \\ &= R\left(\left.\frac{\mathrm{d}\log P}{\mathrm{d}t}\right|_0 t' + \frac{\xi}{\delta t^{1/2}\Delta t}\,\tfrac{1}{2}t'^2\right).\end{aligned} \qquad (27)$$

We need to determine whether the change caused by equation (27) leads to a violation of the constraints at any point during the step. It is sufficient here to consider the endpoints of the step interval, and any turning points that might occur during it. If no constraint violations occur then the saturation change across the full step, which we denote δS, just equals the unconstrained change, and hence

$$\delta S = R\,\delta \log P = R\left(\left.\frac{\mathrm{d}\log P}{\mathrm{d}t}\right|_0 \delta t + \tfrac{1}{2}\frac{\xi\,\delta t^{3/2}}{\Delta t}\right). \qquad (28)$$

If a constraint violation occurs, and there is no turning point in the constrained zone, then the saturation finishes the step on the constraint boundary with either $S = 0$ or $S = 1$ as appropriate. This is indicated in Figure 1a. If a turning point occurs during a constraint violation, then chartist share trading recommences at the turning point. The change in the actual value of S by the end of the step is then the difference between the unconstrained final value and the unconstrained turning point value. This is indicated in Figure 1b.

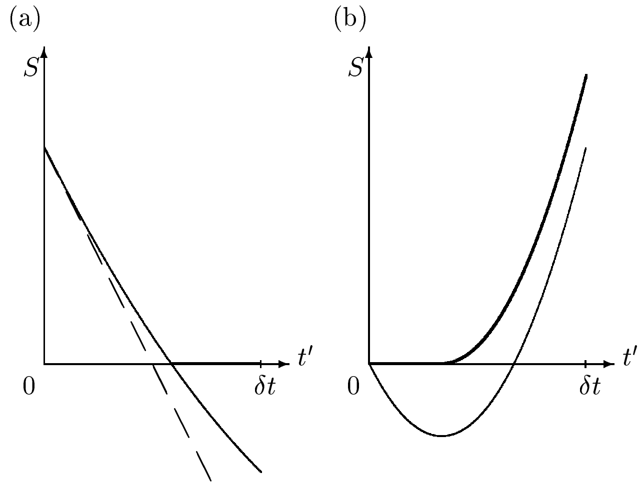

Fig. 1. Chartist saturation S encountering a constraint. (a) The unconstrained S (thin line) falls below zero at some intermediate time between 0 and δt. The constrained S deviates from the unconstrained value after this time, and is fixed at zero for the remainder of the step (thick line). This represents a switching off of the positive feedback in the model, and the initiation of quenching. When determining the switch-off time for the purposes of updating $\mathrm{d}\log P/\mathrm{d}t$ a linear approximation for the unconstrained S (dashed line) is adequate. (b) The unconstrained S (thin line) falls below zero, but has a turning point during the step at some intermediate time between 0 and δt. The constrained S (thick line) is initially held at zero, until after the turning point, when saturation starts increasing, copying the increase of the unconstrained S. This represents a switching on of the positive feedback, which may generate instability. Although the sketches in this figure are specific to the lower constraint boundary $S = 0$, it is obvious that analogous mechanisms apply at the upper constraint boundary $S = 1$.

4.3 Update of d log P/dt

Now we turn to the update of $\mathrm{d}\log P/\mathrm{d}t$, which is the most complicated part of the algorithm, owing to the possible discontinuities that arise when the feedback term switches off or back on.

We begin by differentiating equation (25) to obtain a crude approximation to $\mathrm{d}\log P/\mathrm{d}t$ after an arbitrary time t'

$$\frac{\mathrm{d}\log P}{\mathrm{d}t}^{\mathrm{crude}} = \left.\frac{\mathrm{d}\log P}{\mathrm{d}t}\right|_0 + \frac{\xi\, t'}{\delta t^{1/2}\Delta t}\cdot \qquad (29)$$

Next in the absence of constraints, we substitute equation (4) into equation (6), rearrange and integrate to obtain a formal expression for the change in $\mathrm{d}\log P/\mathrm{d}t$

across a full step, which we denote $\delta(\mathrm{d}\log P/\mathrm{d}t)$

$$\delta\left(\frac{\mathrm{d}\log P}{\mathrm{d}t}\right)=\frac{\xi\,\delta t^{1/2}}{\Delta t}-\frac{1}{\Delta t}\int_0^{\delta t}\frac{\mathrm{d}\log P}{\mathrm{d}t}(t')\,\mathrm{d}t'+\frac{MR}{\Delta t}\int_0^{\delta t}\frac{\mathrm{d}\log P}{\mathrm{d}t}(t')\,\mathrm{d}t'. \qquad (30)$$

The above expression is only formal, because the integrands on the right hand side are not known precisely. However a useful algorithm with the required level of accuracy can be obtained by substituting equation (29) for these right hand side terms, to obtain

$$\delta\left(\frac{\mathrm{d}\log P}{\mathrm{d}t}\right)=\frac{\xi\,\delta t^{1/2}}{\Delta t}-\frac{\delta t}{\Delta t}\left(\left.\frac{\mathrm{d}\log P}{\mathrm{d}t}\right|_0+\tfrac{1}{2}\frac{\xi\,\delta t^{1/2}}{\Delta t}\right)+\frac{MR\,\delta t}{\Delta t}\left(\left.\frac{\mathrm{d}\log P}{\mathrm{d}t}\right|_0+\tfrac{1}{2}\frac{\xi\,\delta t^{1/2}}{\Delta t}\right). \qquad (31)$$

Note the level of error associated with equation (31). We have $O(\delta t/\Delta t)$ relative error in each integrand. Since $\mathrm{d}\log P/\mathrm{d}t$ is itself $O(\Delta t^{-1/2})$, at least in the stable regime, the overall error per step for $\mathrm{d}\log P/\mathrm{d}t$ is $O(\delta t^2\Delta t^{-5/2})$. Velocity errors should persist for roughly one relaxation time $O(\Delta t)$, and hence lead to a per step drift for $\log P$ on the order of $O(\delta t^2\Delta t^{-3/2})$. This is comparable with the error already present in the $\log P$ update, so our update algorithms for $\log P$ and $\mathrm{d}\log P/\mathrm{d}t$ are compatible.

The above discussion has centred on the unconstrained case. If instead the constraints apply over the whole step (*i.e.* no feedback), the update is also simple. Equation (31) essentially applies but with the terms involving MR discarded.

The difficult cases to handle are those where the feedback switches off or back on at some intermediate time during a step. The problem is to determine the time at which the switch occurs to some acceptable order of accuracy. To determine the switch-off point, which we shall denote by δt_{off}, one option is to equate $\delta S^{\mathrm{uncon}}(t')$ given by equation (27), to the distance from the constraint boundary, and then to solve the resulting quadratic equation. A simpler, but less accurate, alternative is to employ a linear approximation using just the values of S and $\mathrm{d}S/\mathrm{d}t$ from the beginning of the step. This is indicated by a tangent line to the actual saturation curve in Figure 1a. This simpler alternative can be justified, owing to the relative rarity of those time steps which actually exhibit a switch-off. Such an argument is certainly valid once the price velocity has attained its $O(\Delta t^{-1/2})$ root mean square value, since velocities are then quite persistent, and the constraint boundary can be crossed at most once per delay time Δt. There is however a possibility that a price will hit a constraint boundary, quench and then dither around, recolliding with the boundary many times, before eventually acquiring its $O(\Delta t^{-1/2})$ velocity and definitively leaving the neighbourhood of the constraint boundary. During the dither phase, price velocities could be much less than $O(\Delta t^{-1/2})$, making them not anywhere near so persistent, and this permits more frequent boundary collisions. However even in these circumstances, it can be shown that collisions occur on average at most once per $O(\Delta t^{1/2}\delta t^{-1/2})$ steps. A decreased price velocity during dither, also implies a poorer than usual estimate of switch-off time δt_{off} if a linear estimate is used as proposed. However this has no bearing on the overall accuracy of the algorithm because the feedback term, which is proportional to $\mathrm{d}S/\mathrm{d}t$ and hence to $\mathrm{d}\log P/\mathrm{d}t$, is itself weak during dither.

It is also necessary to handle those time steps during which a switch-on occurs, denoting the switch-on point by δt_{on} say. Equation (27) can be differentiated with respect to time, to yield

$$\frac{\mathrm{d}\left(\delta S^{\mathrm{uncon}}\right)}{\mathrm{d}t'}=R\left(\left.\frac{\mathrm{d}\log P}{\mathrm{d}t}\right|_0+\frac{\xi t'}{\delta t^{1/2}\Delta t}\right), \qquad (32)$$

which is then equated to zero giving δt_{on} to acceptable accuracy. A better estimate of δt_{on} is not required, because the nature of the switch-on criterion itself guarantees that the $\mathrm{d}S/\mathrm{d}t$ feedback term is relatively unimportant, just as was discussed above in the context of dither.

The general formula for updating $\mathrm{d}\log P/\mathrm{d}t$ for switch-off or -on is therefore

$$\begin{aligned}\delta\left(\frac{\mathrm{d}\log P}{\mathrm{d}t}\right)=\;&\frac{\xi\,\delta t^{1/2}}{\Delta t}\\&-\frac{\delta t}{\Delta t}\left(\left.\frac{\mathrm{d}\log P}{\mathrm{d}t}\right|_0+\tfrac{1}{2}\frac{\xi\delta t^{1/2}}{\Delta t}\right)\\&+\frac{MR}{\Delta t}\left(\left.\frac{\mathrm{d}\log P}{\mathrm{d}t}\right|_0\delta t_{\mathrm{off}}+\frac{\xi}{\delta t^{1/2}\Delta t}\,\tfrac{1}{2}\delta t_{\mathrm{off}}^2\right)\\&+\frac{MR}{\Delta t}\left(\left.\frac{\mathrm{d}\log P}{\mathrm{d}t}\right|_0(\delta t-\delta t_{\mathrm{on}})+\frac{\xi}{\delta t^{1/2}\Delta t}\,\tfrac{1}{2}\left(\delta t^2-\delta t_{\mathrm{on}}^2\right)\right).\end{aligned} \qquad (33)$$

In the above formula we adopt the convention that $\delta t_{\mathrm{off}}=0$ in the case of switch-on alone and $\delta t_{\mathrm{on}}=\delta t$ in the case of switch-off alone.

5 Numerical results

In this section we present numerical results obtained *via* the algorithm of Section 4. We begin by looking at sample market trajectories, focusing on the distinct characteristics of stable and unstable markets. Then we consider mean square and correlation statistics of log price and price velocity, along with distributions of price changes. Next we look at the time taken to cross between boundaries, the time spent dithering around a boundary,

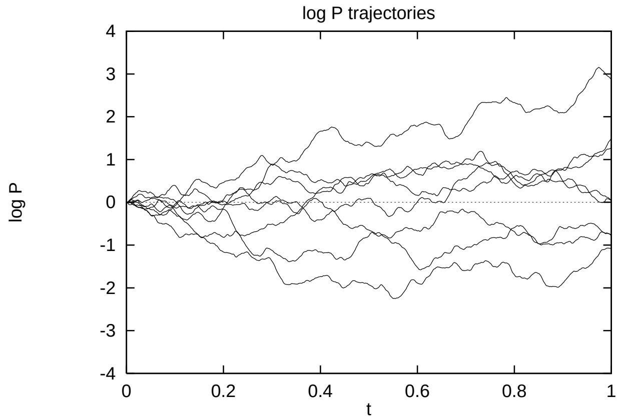

Fig. 2. Sample Langevin random walk trajectories of $\log P$ in a market with delay time $\Delta t = 0.01$, investor responsiveness $R = 1$ and market sensitivity $M = 0$.

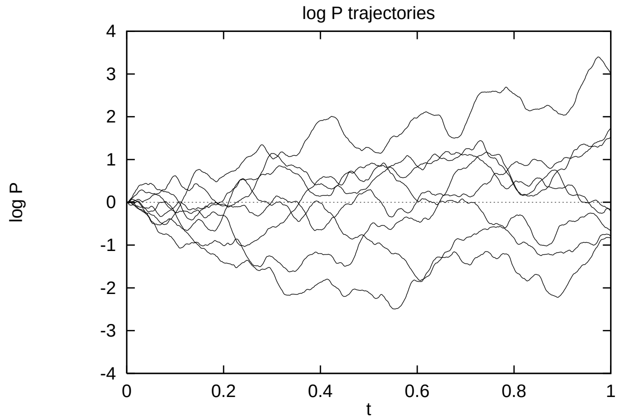

Fig. 3. Sample trajectories of $\log P$ in a market with $\Delta t = 0.01$, $R = 1$ and $M = 0.5$. The sequence of random forces for each trajectory is identical to that in Figure 2. There is a noticeable spreading of the trajectories compared to that figure, but the market remains in a stable regime.

and how saturation values are distributed. Finally we consider a case where chartists sabotage a market, pushing it between stable and unstable regimes at will.

All simulations we present were performed with $\delta t = \frac{1}{10}\Delta t$. Most were performed with $\Delta t = 0.01$. This is considerably larger than the ratio between delay and diffusion times in a real market, but choosing much smaller Δt values, would have required a prohibitively small time step. In qualitative terms, the precise value of Δt is unimportant, provided it is small enough to demonstrate a dramatic separation between delay and diffusion time scales, which is the key feature of real markets we wish to model. We have also chosen $R = 1$ fixed in most simulations, while M has been varied. We took $\log P$ and $\mathrm{d}\log P/\mathrm{d}t$ to vanish at the initial instant, and S to be initially at 50% saturation.

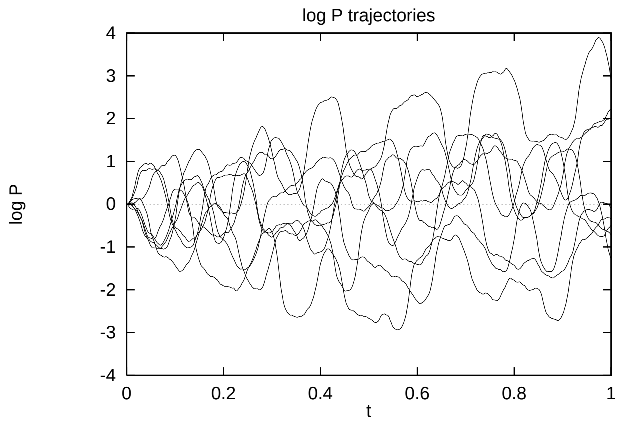

Fig. 4. Sample trajectories of $\log P$ in a market with $\Delta t = 0.01$, $R = 1$ and $M = 1.5$. The sequence of random forces for each trajectory is again identical to that in Figure 2. The trajectories exhibit rapid transitions between upper and lower price boundaries. This is an unstable regime.

5.1 Trajectories of log P and S

Eight sample trajectories of $\log P$ *vs.* t are shown in Figure 2 for the parameter value $M = 0$ with R and Δt as given above. There is no feedback here, and $\log P$ merely executes a random walk, albeit smoothed over a time $\Delta t = 0.01$ and lagging slightly behind the forcing. In Figure 3 we show the trajectories for the same sequence of random forces, but with $M = 0.5$. Qualitatively they are very similar to the trajectories of Figure 2 since the system remains in the stable regime, but there is a detectable increase in the spreading of the trajectories. The diffusivity of the system has been increased by the positive feedback. The value of M is increased further to 1.5 in Figure 4. The system has entered an unstable regime and a major qualitative difference can be seen in the trajectories. The trajectories show rapid up-down transitions interspersed by less steep portions of curve.

It is interesting to consider a single trajectory and to compare the profile of $\log P$ with that of S. We can see in Figure 5, which corresponds to $M = 0$, that changes in S follow those of $\log P$, except that S is constrained to lie between 0 and 1. A fairly long section of curve where $S = 0$ is evident, punctuated by small bumps, which correspond to the system dithering around near the constraint boundary.

Analogous data but with $M = 1.5$ is shown in Figure 6. The S curve now shows numerous perfectly flat portions where the constraints $S = 0$ or $S = 1$ are imposed. Moreover the curvature of the $\log P$ curve changes once S encounters its constraints. This corresponds to a switch from acceleration of $\log P$ due to instability, to deceleration of $\log P$ due to quenching. Some of the constrained sections of the S curve show bumps consistent with dither, but often the system seems to bounce fairly cleanly between the constraint boundaries.

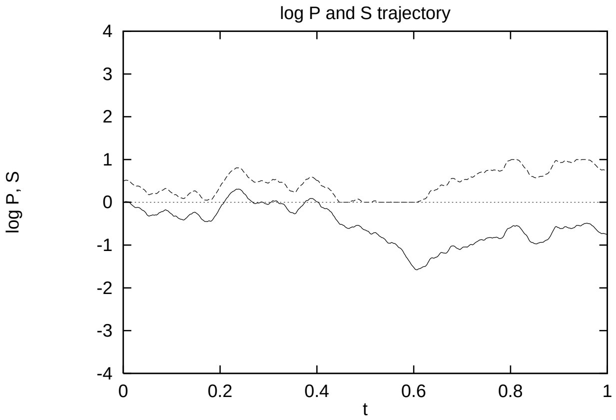

Fig. 5. Sample trajectory of $\log P$ (solid line) and corresponding S (dashed line) for $\Delta t = 0.01$, $R = 1$ and $M = 0$. The changes in S mirror those in $\log P$, but S is subject to the constraints $0 \leq S \leq 1$. The system spends a comparatively small proportion of time on the constraint boundaries, and makes relatively slow transitions between them. This is stable behaviour.

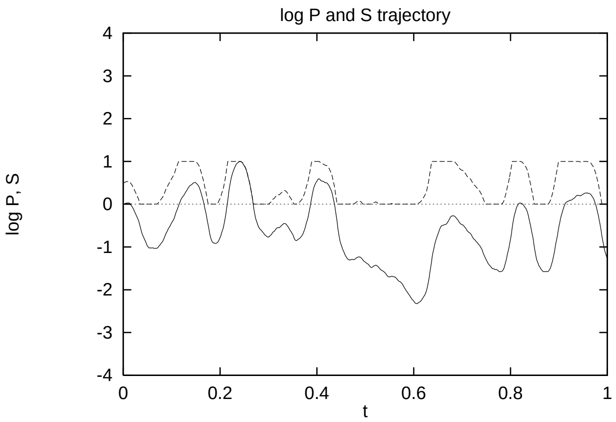

Fig. 6. Sample trajectory of $\log P$ (solid line) and corresponding S (dashed line) for $\Delta t = 0.01$, $R = 1$ and $M = 1.5$. The system spends a comparatively high proportion of time pressed against the constraint boundaries $S = 0$ and $S = 1$, and bounces extremely rapidly between them. This is unstable behaviour.

5.2 Statistics of log P and d log P/dt

Although we can gain useful intuition from considering individual trajectories, there is always the risk that the trajectory we select may be in some way atypical. Thus it is important also to compile statistics over many trajectories.

A useful statistic is the variance of $\log P$ about its initial value $\log P_0$. Figure 7 shows a plot of $\langle(\log P - \log P_0)^2\rangle$ *vs.* time, for various M, still with $R = 1$ and $\Delta t = 0.01$. The data have been obtained by averaging over 16 384 trajectories. For $M = 0$ the variance grows linearly in t when $t \gg \Delta t$. This agrees with the case of pure geometric Brownian motion and with the asymptotic form given in equation (8). Furthermore a theoretical formula is

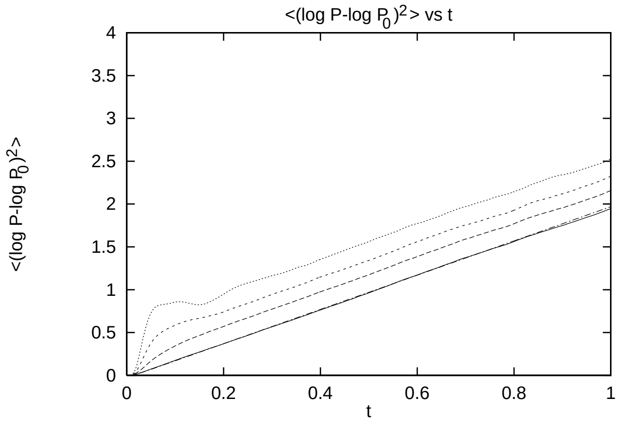

Fig. 7. Statistics of $\langle(\log P - \log P_0)^2\rangle$ *vs.* t taken over 16 384 realizations with $\Delta t = 0.01$ and $R = 1$ in all cases, and $M = 0$ (solid line), $M = 0.5$ (long dashes), $M = 1$ (short dashes) and $M = 1.5$ (dotted line). A theoretical result for $M = 0$ is also shown (dash-dot line) which, after a short lag, grows like $2t$ for long times, and this is in good agreement with data. The larger MR values (unstable cases) show $\langle(\log P - \log P_0)^2\rangle$ increasing dramatically initially, but this effect is short lived, and thereafter the rate of increase is the same for all MR values.

available for $M = 0$ over the full range of t. This is shown on the graph, and is in good agreement with data.

As M increases in Figure 7 the curves show increasingly large initial bursts of growth. However these appear to be extremely short lived, typically only a few multiples of Δt, after which the constraints must make their presence felt. Thus the unconstrained unstable runaway growth described by equation (12) only lasts a very short time. The surprising feature about Figure 7 is that after these initial bursts, the slope of each curve is identical to the $M = 0$ case. Thus while short lived booms and crashes may benefit or harm individual investors in the short term, the quenching mechanism that we propose is very effective at limiting excessive changes in the market over the long term.

Note that for the unstable case ($MR = 1.5$), there is a definite oscillation in $\langle(\log P - \log P_0)^2\rangle$ before it settles down to its straight line growth. The minimum of this oscillation could correspond to trajectories passing back close to the initial log price, after hitting the constraint boundary once and bouncing off it. It should be remembered that Figure 7 represents an average over many trajectories. Different trajectories will encounter the constraints at different times, but nonetheless it can be sensible to consider a typical time for the system to hit the boundary and bounce back. However, after several bounces the uncertainty in the state of the system will become too great to allow one to designate any single trajectory as typical. Thus the oscillatory behaviour is rapidly damped. These damped oscillations are reminiscent of the spatial oscillations one encounters in statistical mechanics, when one is calculating the pair correlation function for *packing* a number of *objects*, *e.g.* hard spheres, into

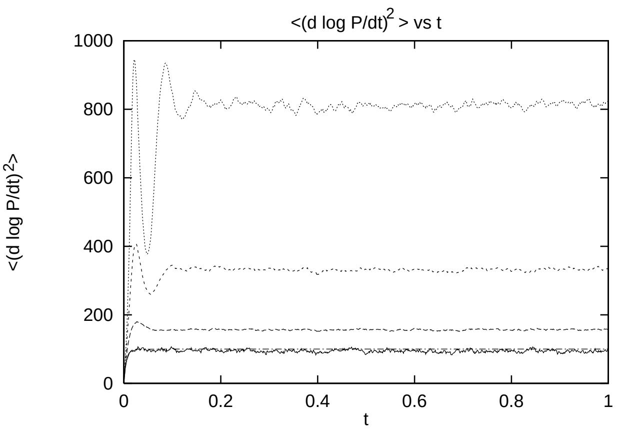

Fig. 8. Statistics of $\langle(\mathrm{d}\log P/\mathrm{d}t)^2\rangle$ *vs.* t taken over 16 384 realizations with $\Delta t = 0.01$ and $R = 1$ in all cases, and $M = 0$ (solid line), $M = 0.5$ (long dashes), $M = 1$ (short dashes) and $M = 1.5$ (dotted line). By assumption $\mathrm{d}\log P/\mathrm{d}t$ vanishes at $t = 0$. A theoretical result for $M = 0$ is also shown (dash-dot line) approaching Δt^{-1} for long times, and this is in good agreement with data. In the unstable cases (large MR), an asymptotic mean square velocity, albeit at a somewhat higher level, is also achieved. The quenching mechanism is extremely effective at limiting unbounded increase of $\langle(\mathrm{d}\log P/\mathrm{d}t)^2\rangle$.

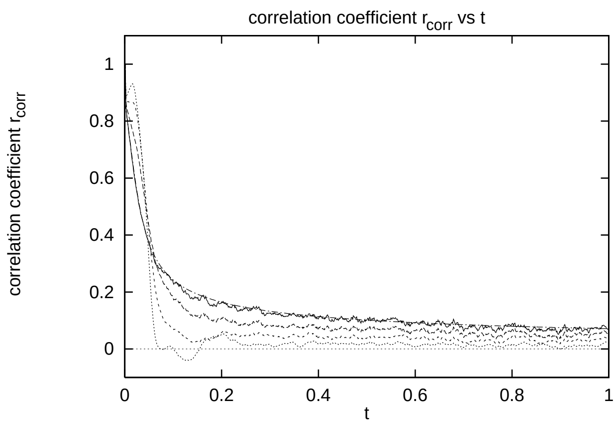

Fig. 9. The correlation coefficient r_{corr} *vs.* t taken over 16 384 realizations with $\Delta t = 0.01$ and $R = 1$ in all cases, and $M = 0$ (solid line), $M = 0.5$ (long dashes), $M = 1$ (short dashes) and $M = 1.5$ (dotted line). By assumption $\mathrm{d}\log P/\mathrm{d}t$ vanishes at $t = 0$. A theoretical result for $M = 0$ is also shown (dash-dot line), in good agreement with data. This decays as $(\Delta t/2t)^{1/2}$, because $\log P - \log P_0$ depends on the entire forcing history, whereas $\mathrm{d}\log P/\mathrm{d}t$ only depends on forcing over an $O(\Delta t)$ interval in the recent past, *i.e.* there is little correlation between the log price itself and the price velocity. The largest MR value (unstable case) shows an initial increase in correlation between $\log P - \log P_0$ and $\mathrm{d}\log P/\mathrm{d}t$ as the initial random forcing fixes the direction of evolution. However this decays rapidly as the system first hits the constraints. The unstable system thereafter makes rapid transitions between the constraint boundaries, so $\mathrm{d}\log P/\mathrm{d}t$ fluctuates in sign, and is poorly correlated with $\log P - \log P_0$.

a given *space* [38,39]. The analogy arises because here we are attempting to *schedule* a number of *events* (bounces) within a given *time*.

In Figure 8 we show the mean square of the price velocity $\langle(\mathrm{d}\log P/\mathrm{d}t)^2\rangle$ for the same set of parameter values as Figure 7. For simplicity, $\mathrm{d}\log P/\mathrm{d}t$ has been assumed to vanish at the initial instant. There is a final asymptotic mean square velocity, in all cases. For $M = 0$ a theoretical prediction is given of $\langle(\mathrm{d}\log P/\mathrm{d}t)^2\rangle$ which is in good agreement with data, and which agrees with the asymptotic prediction of equation (9) in the long time limit.

The final asymptotic mean square velocity is an increasing function of M. For our data the increase is a factor of roughly ten when M changes from 0 to 1.5. If the heuristic arguments of Section 3.4 are to be believed, the mean square velocity would increase only very modestly if the unstable regime were dominated by dither, but quite dramatically by $O(\Delta t^{-1})$ if the unstable regime were dominated by crossings. The actual case is somewhere between these two extremes.

Finally in Figure 9 we show the correlation coefficient (between log price and price velocity) r_{corr} *vs.* time. For $M = 0$ this decays proportional to $(\Delta t/t)^{-1/2}$ when $t \gg \Delta t$ as per equation (10). There is good agreement between theory and data. For the unstable system with $M = 1.5$, the correlation initially grows with time as early forcing selects and fixes a direction of price change. However once the constraint boundaries are encountered, the correlation crashes down to zero. There is even a brief period of negative correlation related to the oscillations noted earlier for Figure 7. Subsequent to this the price is bouncing rapidly between the constraint boundaries. Knowing its value, is therefore no help in knowing its direction of movement.

5.3 Statistics of returns

One potential disadvantage with the variance statistics considered in the previous subsection is that they can depend on the special initial conditions chosen, in our case $\log P = \mathrm{d}\log P/\mathrm{d}t = 0$. It is desirable to consider a different statistic which does not suffer from this problem. One such statistic is the "variance of $\log P$ returns". We define a return to be the difference in log price across a specified time interval, which is called the return time and denoted t_{ret}. Although the return compares the log price at two instants separated in time, it does not refer the state of the market back to any arbitrarily chosen initial condition, since we can always follow a trajectory for long enough to ensure that the initial conditions will be forgotten.

Statistics of returns were accumulated by tracking a single trajectory over a total dimensionless time $t = 20\,000$. We have considered return times t_{ret} of 0.001 through 1.024 in powers of 2. This set of t_{ret} values brackets $\Delta t = 0.01$, which should ensure that an interesting range of behaviours will be considered. Throughout we set $R = 1.0$, and we treat the cases $M = 0.0$, $M = 0.5$, $M = 1.0$ and $M = 1.5$.

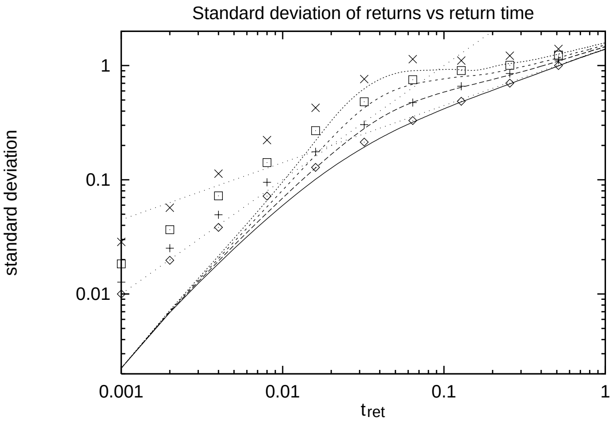

Fig. 10. Standard deviation of returns *vs.* return time, for $\Delta t = 0.01$ and $R = 1.0$, and $M = 0.0$ ($\Diamond$), $M = 0.5$ ($+$), $M = 1.0$ ($\boxdot$) and $M = 1.5$ ($\times$). Statistics are accumulated over 20 000 dimensionless time units. The dotted lines indicate $\Delta t^{-1/2} t_{\rm ret}$ and $(2t_{\rm ret})^{1/2}$ which are the asymptotic forms applying (for $M = 0.0$) at short and long return times respectively. For comparison purposes we also show the time evolution of standard deviation of log price about its initial value, for $M = 0.0$ (solid line), $M = 0.5$ (long dashes), $M = 1.0$ (short dashes) and $M = 1.5$ (very short dashes). At early times, these lie far below the corresponding standard deviations of returns, since the chosen initial condition makes the price slow moving at first. However at longer times once the initial conditions are forgotten there is good agreement.

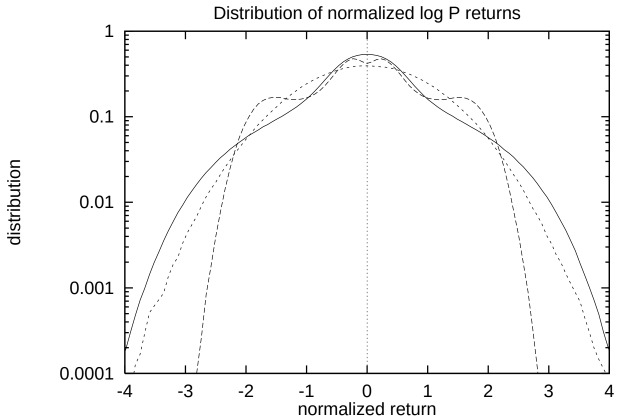

Fig. 11. Distributions of normalized returns for $M = 1.5$, $R = 1.0$ and $\Delta t = 0.01$, and return times $t_{\rm ret}$ of 0.001 (solid line), 0.032 (long dashes) and 1.024 (short dashes). For $t_{\rm ret} = 0.001$ the normalized distribution has a central peak plus broad shoulders. For $t_{\rm ret} = 0.032$ the distribution exhibits very prominent wings, while for $t_{\rm ret} = 1.024$ it has a relatively Gaussian appearance. All data have been accumulated over 20 000 dimensionless time units.

Results for the standard deviations of returns are plotted in Figure 10. For $M = 0.0$ the returns show a clear transition from deterministic behaviour for $t_{\rm ret} \ll \Delta t$, to diffusive behaviour for $t_{\rm ret} \gg \Delta t$. The standard deviation of returns asymptotes to $\Delta t^{-1/2} t_{\rm ret}$ for small $t_{\rm ret}$, corresponding to a log price evolving deterministically at the equilibrium root mean square speed $\Delta t^{-1/2}$. For large $t_{\rm ret}$ the standard deviation of returns is $(2t_{\rm ret})^{1/2}$. This corresponds to log price executing a random walk with unit diffusivity.

As M increases the standard deviation of returns also increases for given $t_{\rm ret}$. However this increase ceases to be important for $t_{\rm ret}$ values bigger than a few multiples of Δt, *i.e.* market instability is of little importance on long time scales. In the unstable regime with $MR = 1.5$ the standard deviation of returns appears to be non-monotonic, with that for $t_{\rm ret} = 0.128$ being less than that for $t_{\rm ret} = 0.064$. One expects the local minimum in the standard deviation of returns corresponds to the time required for a typical trajectory to cross from one boundary to the other and back again.

In Figure 10 we have also plotted for comparison the standard deviation of the log price about its initial position (this is simply the square root of the data in Fig. 7). At times long compared to Δt there is good agreement between the standard deviation of returns and the standard deviation about the initial value. This is because at long times the arbitrary initial price velocity is forgotten. However at short times on the order of Δt or less, the standard deviation about the initial value is much smaller than that of the returns: our chosen initial condition $\log P = {\rm d}\log P/{\rm d}t = 0$ causes the price to be slow to start moving. Recall also that for $MR = 1.5$, the standard deviation of $\log P$ about the initial location shows damped oscillations with time, and these correlate well with the non-monotonic behaviour of the standard deviation of returns.

5.4 Distributions of returns

In the previous subsection we considered one statistic of the returns, the standard deviation. Since the mean return vanishes here (no mean market growth or interest rates in the model), the standard deviation would be sufficient to characterize completely a *Gaussian* distribution of returns. However it is well known in financial systems [9,21–29] that the distribution of returns is non-Gaussian, in particular the tails of the distribution should have a much higher weight than a Gaussian would permit. In the absence of feedback $M = 0.0$, the model we consider reduces to a linear Langevin equation, which can be shown to give Gaussian behaviour. We ask therefore whether a non-zero value of M can produce a non-Gaussian distribution. For simplicity we consider here only the value $M = 1.5$ which has strong feedback (along with $R = 1.0$ and $\Delta t = 0.01$ as before). Again statistics are accumulated over a dimensionless time of $t = 20\,000$.

In Figure 11 we have plotted the distributions at $t_{\rm ret} = 0.001$, $t_{\rm ret} = 0.032$ and $t_{\rm ret} = 1.024$. These distributions have been normalized by their standard deviations, so that we can focus more closely upon their shape, rather than merely their width. For the largest return time considered $t_{\rm ret} = 1.024 \gg \Delta t$, the distribution is relatively Gaussian in appearance. However for the smallest value

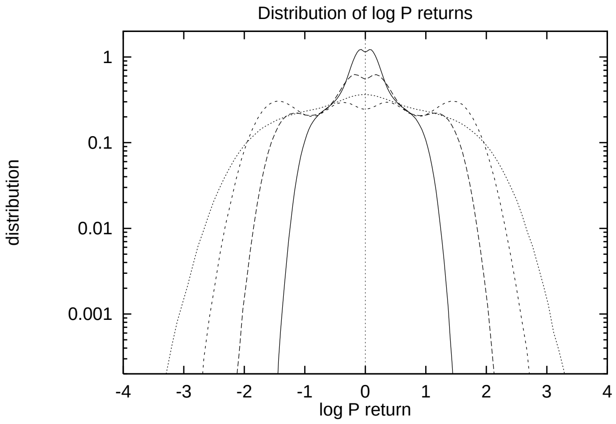

Fig. 12. Distributions of returns (unnormalized) for $M = 1.5$, $R = 1.0$ and $\Delta t = 0.01$, with returns accumulated over 20 000 dimensionless time units, and with return times $t_{\mathrm{ret}} = 0.016$ (solid line), $t_{\mathrm{ret}} = 0.032$ (long dashes), $t_{\mathrm{ret}} = 0.064$ (short dashes) and $t_{\mathrm{ret}} = 0.128$ (very short dashes). For $t_{\mathrm{ret}} = 0.016$ the distribution has a central peak with shoulders. As t_{ret} increases to 0.032 the shoulders broaden into wings, while for $t_{\mathrm{ret}} = 0.064$, the wings are very prominent. Here the wings occur for returns of roughly ± 1.5 corresponding to the system crossing from one constraint boundary to the other ($\log P$ changes by unity), plus some overshoot of $\log P$ during quenching. For $t_{\mathrm{ret}} = 0.128$ the distribution is starting to tend to a Gaussian shape.

$t_{\mathrm{ret}} = 0.001 \ll \Delta t$, the distribution is non-Gaussian. It has a central peak, surrounded by fairly broad shoulders, finally merging into the tails. Thus there is a population of small magnitude returns giving the peak, and a population of larger magnitude returns giving the shoulders. In spite of this additional structure, the distribution still falls off quite rapidly (like a Gaussian) in the tails.

For an intermediate $t_{\mathrm{ret}} = 0.032$, the shoulders of the distribution develop into wings, *i.e.* the distribution develops secondary peaks away from a zero return. The distribution tails off rapidly beyond these wings.

In order to understand the significance of the wings we have plotted in Figure 12 several more distributions for times t_{ret} of 0.016, 0.032, 0.064 and 0.128. We have not normalized these distributions by the standard deviation here, which means they become broader as t_{ret} increases.

For $t_{\mathrm{ret}} = 0.016$ the distribution is still qualitatively rather like that in Figure 11 for a very short return time $t_{\mathrm{ret}} = 0.001$, *i.e.* it has a central peak and shoulders. However as the return time increases in Figure 12 to $t_{\mathrm{ret}} = 0.032$ the wings develop, and they become very prominent for $t_{\mathrm{ret}} = 0.064$.

The advantage of plotting distributions of unnormalized returns here is that it becomes immediately apparent that the peaks of the wings are for returns, *i.e.* changes in $\log P$, of around ± 1.5. Returns of this magnitude must correspond to S swinging between the constraint boundaries at zero and unity, during which period $\log P$ would change by precisely ± 1 for $R = 1.0$. After S hits the constraint, $\log P$ must overshoot during quenching, so that the overall change in $\log P$ is somewhat larger than unity. For yet longer return times $t_{\mathrm{ret}} = 0.128$ or greater, we are no longer looking at a single upswing or downswing, but rather the cumulative effect of two or more such consecutive swings. There is no longer a typical $\log P$ return of ± 1.5, so the wings disappear, and overall the distribution broadens. The distribution for $t_{\mathrm{ret}} = 0.128$ is tending toward a Gaussian. Its shape is not too different from the $t_{\mathrm{ret}} = 1.024$ distribution in Figure 11.

Whilst it is encouraging that overall non-Gaussian distributions can be produced by the model considered, the tails of the distributions in Figures 11, 12 still fall off very rapidly, just like Gaussian tails. Therefore the model in its current form does not capture all features of a real market for which we have stated the tails should receive a much higher weight [9,21–29]. It is interesting to consider what changes would be required in the model to reproduce the observed tail weightings. Lux and Marchesi [9] have considered a model in which individual investors can either adopt or abandon speculating strategies according to the state of the market. They find that high volatility bursts produce the non-Gaussian tails and that these bursts are associated with an abnormally high number of investors behaving as speculators. In the context of the present Langevin type model this would correspond to coupling the responsiveness R to the state of the market $\log P$ and $\mathrm{d}\log P/\mathrm{d}t$. Note that one possible type of coupling has already been proposed *via* the risk averse term of Bouchaud and Cont [30]. However it is not clear whether such a term by itself can reproduce the required tail weightings for the distributions. Other aspects of investor behaviour may need to be modelled too. For instance, if the market is evolving rapidly (large $\mathrm{d}\log P/\mathrm{d}t$), and many high profile players are making quick profits, the associated publicity may well induce others to play the market, leading to an increase in R. We do not pursue such extensions to the model here.

5.5 Crossing and dither time

It is desirable to find suitable statistics that clearly show the change in governing time scale from an $O(1)$ scale to an $O(\Delta t)$ scale as the system enters the instability regime.

We have chosen two suitable statistics called crossing time and dither time which we define precisely as follows. Crossing time is the transit time from one boundary to the other, conditional on not returning to the original boundary. Each crossing is followed by a short quenching phase during which $\mathrm{d}\log P/\mathrm{d}t$ falls to zero. The quenching phase duration is always on the order of Δt, irrespective of the stability regime. In practice more unstable systems take very slightly longer to quench since they exhibit more violent collisions with the boundaries. The dither time is the time taken from the end of quenching to the start of the next crossing. The system may leave and return to the boundary several times during the dither phase, conditional on it not reaching the opposite boundary.

Figure 13 shows a plot of crossing and dither time for $R = 1$, $\Delta t = 0.01$ and M varying. The curves represent

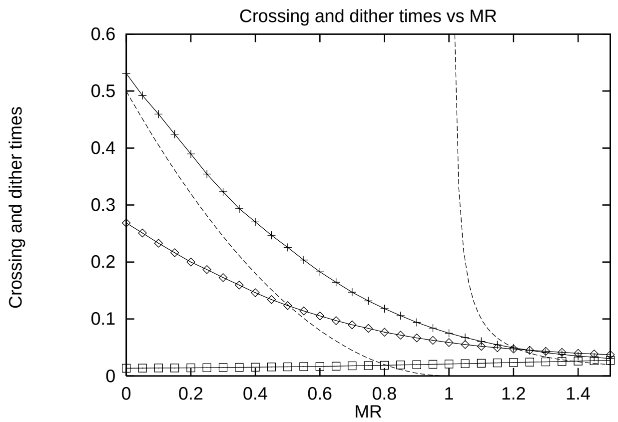

Fig. 13. Average crossing time between the constraint boundaries (◇), average dithering time near either constraint boundary (+), and average quenching time after hitting a boundary (□) as functions of MR. These data are averaged over 10 000 crossing events and correspond to $\Delta t = 0.01$, $R = 1$ and M varying. As MR increases, the transitions themselves occupy less time, and since even small fluctuations can nucleate a crossing event, correspondingly less time is spent dithering. The quenching time is $O(\Delta t)$ in all cases, but increases slowly with MR as a result of more violent collisions. Rough estimates of the crossing time $\frac{1}{2}(1-MR)^2R^{-2}$ (stable regime) and $\Delta t\,(MR-1)^{-1}$ (unstable regime) given by equations (18, 19) agree only qualitatively with data (dashed lines).

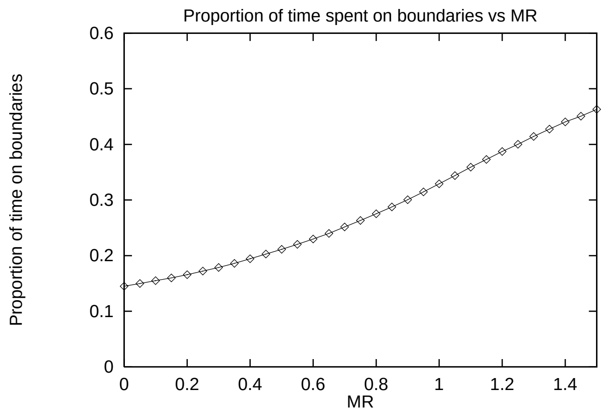

Fig. 14. Proportion of time spent on the constraint boundary as a function of MR, again for $\Delta t = 0.01$, $R = 1$, M varying and averaged over 10 000 crossing events. As the unstable regime is approached and the crossing time (off the boundary) and dither time (partly on the boundary) both fall, the quenching phase (invariably on the boundary) becomes an increasingly important component of the dynamics.

an average over 10 000 crossings for each M. Clearly there is a dramatic decrease in both crossing and dither as the instability regime is approached. For larger MR the system finds it easier both to cross between the boundaries, and to receive a fluctuation permitting it to leave a boundary. Dither time is about twice as large as crossing time in the stable regime, but roughly the same as crossing time for the most unstable case considered $MR = 1.5$. Also shown in the figure, are heuristic guesses of how the crossing time should vary based on equations (18, 19). These show only qualitative agreement with data.

The system is pressed against the constraint boundaries during the entire quenching phase and during part of the dither phase. As the crossing and dither time fall, and become comparable with the $O(\Delta t)$ quenching time, it is reasonable to suppose that the system might spend a higher proportion of its time on the boundary. That this is the case can be seen in Figure 14. When $MR = 1.5$ the system spends nearly half its time on a boundary, which is a factor of three greater than the corresponding proportion when $MR = 0$.

5.6 Histograms of S

In Figure 15 we show a histogram of saturation S values for the parameters $M = 0$, $R = 1$ and $\Delta t = 0.01$. The histograms have been binned over a total time t of 20 000, corresponding to 2×10^7 steps. For clarity points actually on the constraint boundaries have been discarded, otherwise they would give excessive weight to the terminal bins.

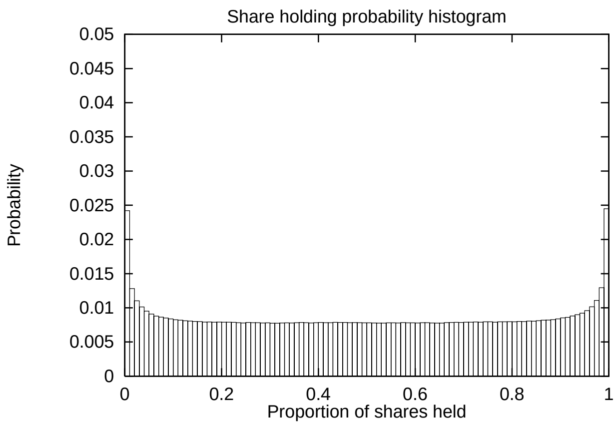

Fig. 15. Probability histogram binned over proportion of shares held for $\Delta t = 0.01$, $R = 1$, $M = 0$, and a total time of 20 000. For clarity, data actually on the constraint boundaries are not indicated, so the sum of the probabilities shown is strictly less than unity.

The histogram is remarkably flat over a wide range of S values, which is consistent with a random walk with no preferred position. There is of course an accumulation of points near $S = 0$ and $S = 1$ owing to the presence of the constraints. However only points within $O(\Delta t^{1/2})$ of the constraint boundaries are affected by this accumulation, because this distance corresponds to an $O(\Delta t^{-1/2})$ root mean square velocity persisting over an $O(\Delta t)$ correlation time.

A corresponding histogram for $M = 1.5$ (unstable case) is shown in Figure 16. The histogram is no longer so flat. This is because the system spends a high proportion of time dithering near the constraint boundaries, but tends to pick up speed as it executes a crossing. Very little time is therefore spent in the centre of the domain.

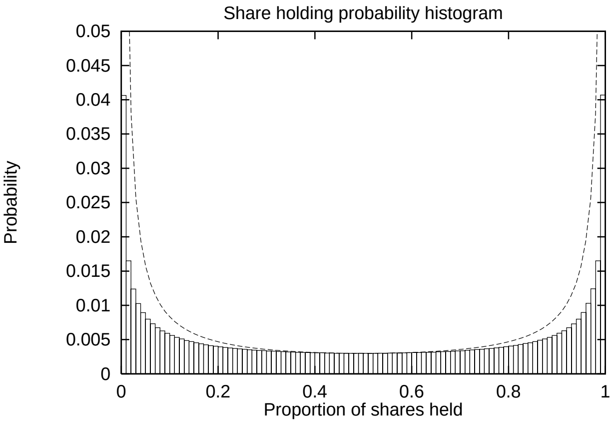

Fig. 16. Probability histogram binned over proportion of shares held for $\Delta t = 0.01$, $R = 1$, $M = 1.5$, and a total time of 20 000. For clarity, data actually on the constraint boundaries are not indicated, so the sum of the probabilities shown is strictly less than unity. Owing to the rapid transitions in this unstable regime, which actually accelerate during the crossing, proportionately less time is spent far from the constraint boundaries than was the case for $MR = 0$. Also shown (dashed line) is a trial functional form for the probability distribution from equation (23). This is the form we expect for trajectories accelerating exponentially in time, and it agrees only qualitatively with data.

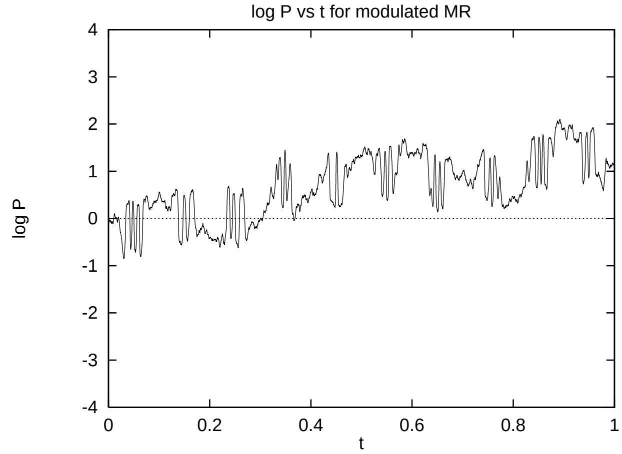

Fig. 17. A sample $\log P$ trajectory where investors are sabotaging the market by modulating their responsiveness R. The parameter values are $\Delta t = 0.0005$ and $M = 1$, while R varies sinusoidally between 0.1 and 1.5 with oscillation period 0.1. Bursts of rapid random activity are seen near the instantaneous maxima of MR, with quiescent phases between them.

The functional form suggested by equation (22) is included on the graph, scaled to the correct value at the centre. It shows only qualitative agreement with the data.

5.7 Market sabotage

The stability-instability behaviour of the model allows the intriguing possibility of investors forcing the market from a stable to an unstable regime by modulating their R value. This could be viewed as a form of economic sabotage.

In order to distinguish clearly between phenomena on the diffusion and delay time scales, we have made Δt very small here, $\Delta t = 0.0005$. Since we have retained $\delta t = \frac{1}{10}\Delta t$, simulation is expensive, but as we are interested here in a single sample trajectory, rather than obtaining statistics over many trajectories, the computation continues to be feasible. We have fixed $M = 1$ here, and allowed R to vary sinusoidally from 0.1 to 1.5 with oscillation period 0.1.

The results are shown in Figure 17. It is clearly seen that the market exhibits quiescent phases during which only slow changes take place, followed by noisy phases during which rapid bursts of activity occur. The bursts happen at regular intervals corresponding to the maxima of MR.

The oscillation period has been deliberately chosen here to be intermediate between the $O(1)$ diffusion time and the $O(\Delta t)$ delay time. This ensures that the market changes comparatively little between bursts, but bounces between the constraint boundaries many times during an individual burst. It is permissible to think of the model crudely as being a two state system, with the states being a high price state (saturation near unity) and a low price state (saturation near zero). The state is completely randomized by the bursts, and afterwards is more or less frozen till the next burst. We have effectively a Markov chain of states. This phenomenon is rather reminiscent of stochastic resonance in physics [40,41], for which modulation and noise couple nonlinearly to induce transitions in a bistable system.

The ability to perform sabotage is of limited use in the present model, since the saboteurs cannot really predict whether the market will boom or crash as a result of their actions, only that it will become extremely unpredictable. However if a risk aversion term were included in the model [30], the saboteurs could be fairly confident of engineering a crash.

The sort of information considered above may also be of use to a central bank contemplating intervention to stabilize a market. The stability criterion $MR = 1$ provides a quantitative measure of the appropriate level of intervention, *i.e.* the size of the negative contribution to R that would be required.

6 Concluding remarks

The Langevin model for asset price can exhibit both stable and unstable regimes, depending on the market sensitivity to investor activity and on the investor response to price change. The stable-unstable transition is interesting because of the large difference in time scale between the slow stable behaviour (governed by the market volatility in the absence of feedback, typically on the order of years) and the rapid unstable behaviour (governed by the delay time, and on the order of minutes). Even when the short term unstable behaviour is present, a stabilizing influence operates *via* a quenching mechanism when the market becomes either deserted or saturated by chartists. In spite of the opportunities for individuals to make a quick profit (or loss) owing to the fast instabilities, the quenching mechanism is very effective in the long term, and the market behaviour is quite robust, being very little different from that in the absence of feedback.

The model studied in the present paper is a deliberately simplified one: our aim here was to formulate a crude description containing the crucial ingredients of delay, feedback and quenching. It is worthwhile speculating how the model behaviour might be altered when more realistic effects are introduced. The three refinements we discuss briefly below are

1. investor delay,
2. mean market growth/interest rates, and
3. profit taking/corporate raiding.

6.1 Investor delay

If an investor delay $\Delta t_{\rm i}$ is added to the response on top of the market delay Δt, it can be shown that the coupled price-saturation equations analogous to equations (2, 5) have two eigenvalues. One is always negative, and the other is negative if and only if $MR < 1$, regardless of the ratio $\Delta t_{\rm i}/\Delta t$. Thus the boundary between stable and unstable behaviour remains at $MR = 1$, but the actual growth and decay rates become sensitive to investor delay. In the limit $\Delta t_{\rm i}/\Delta t \ll 1$, and in the stable regime, decay rates are $1/\Delta t_{\rm i}$ and $\Lambda_{\rm s}/\Delta t$ respectively. The less stable rate is $\Lambda_{\rm s}/\Delta t$ so the results of the present paper are regained. In the opposite limit $\Delta t_{\rm i}/\Delta t \gg 1$, the rates become $\Lambda_{\rm s}/\Delta t_{\rm i}$ and $1/\Delta t$, the former now being the less stable. In the unstable regime, the less stable rate switches sign from decay to growth. These rates follow identical formulae, but with $\Lambda_{\rm u}$ replacing $\Lambda_{\rm s}$.

6.2 Mean market growth/Interest rates

Suppose that a constant bias ν drives the asset price in addition to the random noise Ξ. Further suppose an interest rate $\rho < \nu$ is available on riskless investments, making the (risky) market attractive to chartists only when $\mathrm{d}\log P/\mathrm{d}t > \rho$. In the absence of feedback, $\log P$ grows at a mean rate ν, while feedback effects can be shown to increase this to $\nu + (1 - \Lambda_{\rm s})(\nu - \rho)/\Lambda_{\rm s}$. For long enough times, mean growth (linear in time) will always dominate the spread of log price (square root with time). In other words investors are almost certain to profit from shares provided they retain these investments long enough.

In the unstable regime $MR > 1$, mean market growth introduces an asymmetry which forces the system against the upper constraint boundary in preference to the lower. However crashes will not be eliminated altogether, because the mean growth rate ν is unlikely to dominate the price velocity fluctuations (scaling like $\Delta t^{-1/2}$) that cause transition. Moreover non-zero interest rates tend to reduce the upper-lower asymmetry slightly. Indeed there is a heightened sensitivity to interest rates in the unstable regime: ν/ρ must exceed MR, not merely unity, if the market is to grow on average. However the question of mean market growth is somewhat irrelevant here, since it may be shown that the spread of $\log P$ values vastly exceeds the change in the mean by a large factor on the order of $\Delta t^{-1/2}$. Thus in the unstable regime, the only "prediction" we can reliably make, is that the market very rapidly becomes unpredictable! This shift from a sure profit to a very risky regime occurs because investors can no longer detect the underlying mean growth beneath the magnified noise [18].

6.3 Profit taking/Corporate raiding

The quenching mechanism we have considered here is that the chartists either saturate the market (during a boom) or desert it altogether (during a crash). In real markets, investors may decide to sell during a boom, if the asset price has risen sufficiently above the purchase price (profit taking). Likewise an asset whose share price is crashing, may become attractive to a corporate raider hoping to strip the assets of the company. These are examples of fundamentalist rather than chartist behaviour, which should stabilize the market. However as the market exhibited long term stability even with the comparatively weak quenching mechanism studied in this paper, the long term effects of profit taking/corporate raiding are expected to be minimal. These effects would however be needed to describe a sequence of mini-booms or mini-crashes following one after the other, instead of the alternate booms and crashes we have found here.

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References

1. L. Bachelier, *Théorie de la Spéculation* (Gauthier-Villars, Paris, 1900).
2. L. Bachelier, in *The Random Character of Stock Market Prices*, edited by P.H. Cootner (The MIT Press, Cambridge, Mass., 1964), pp. 17–78, English translation.
3. M.F.M. Osborne, in *The Random Character of Stock Market Prices*, edited by P.H. Cootner (The MIT Press, Cambridge, Mass., 1964), pp. 100–128. Strictly Bachelier's work considered only a normal distribution of price. The modern approach, following Osborne, is to use a log-normal distribution and hence geometric Brownian motion.
4. F. Black, M. Scholes, J. Political Econ. **81**, 637 (1973).
5. H.E. Stanley, L.A.N. Amaral, D. Canning, P. Gopikrishnan, Y. Lee, Y. Liu, Physica A **269**, 156 (1999).
6. R.N. Mantegna, H.E. Stanley, *An Introduction to Econophysics: Correlations and Complexity in Finance* (CUP, Cambridge, 1999).

7. D. Stauffer, Amer. J. Phys. **67**, 1207 (1999).
8. H. Lévy, M. Lévy, S. Solomon, *Microscopic Simulation of Financial Markets* (Academic Press, New York, 2000).
9. T. Lux, M. Marchesi, Nature **397**, 498 (1999).
10. E. Egenter, T. Lux, D. Stauffer, Physica A **268**, 250 (1999).
11. R.G. Palmer, W.B. Arthur, J.H. Holland, B. LeBaron, P. Tayler, Phys. D **75**, 264 (1994).
12. M. Lévy, H. Lévy, S. Solomon, J. Phys. I France **5**, 1087 (1995).
13. P. Bak, M. Paczuski, M. Shubik, Physica A **246**, 430 (1997).
14. G. Caldarelli, M. Marsili, Y.C. Zhang, Europhys. Lett. **40**, 479 (1997).
15. J.P. Bouchaud, R. Cont, Herd behaviour and aggregate fluctuations in financial markets (1997); `http://xxx.lanl.gov/abs/cond-mat/9712318`.
16. D. Stauffer, D. Sornette, Physica A **271**, 496 (1999).
17. D. Chowdhury, D. Stauffer, Eur. Phys. J. B **8**, 477 (1999).
18. F. Slanina, Y.C. Zhang, Physica A **272**, 257 (1999).
19. C. Bußhaus, H. Rieger, Physica A **267**, 443 (1999).
20. G. Iori, Int. J. Mod. Phys. C **10**, 1149 (1999).
21. B.B. Mandelbrot, J. Business **36**, 394 (1963).
22. B.B. Mandelbrot, J. Business **40**, 393 (1967).
23. P. Gopikrishnan, M. Meyer, L.A.N. Amaral, H.E. Stanley, Eur. Phys. J. B **3**, 139 (1998).
24. P. Gopikrishnan, V. Plerou, L.A.N. Amaral, M. Meyer, H.E. Stanley, Phys. Rev. E **60**, 5305 (1999).
25. J. Laherrère, D. Sornette, Eur. Phys. J. B **2**, 525 (1998).
26. R. Mantegna, H.E. Stanley, Phys. Rev. Lett. **73**, 2946 (1994).
27. R. Mantegna, H.E. Stanley, Nature **376**, 46 (1995).
28. J.P. Bouchaud, M. Potters, *Théorie des Risques Financiers: Portefeuilles, Options et Risques Majeurs*, number 6 in Aléa-Saclay (Commisariat à l'énergie atomique, Paris, 1997), in French.
29. J.P. Bouchaud, Physica A **23**, 415 (1999).
30. J.P. Bouchaud, R. Cont, Eur. Phys. J. B **6**, 543 (1998).
31. F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, Singapore, 1965).
32. D.T. Gillespie, *Markov Processes: An Introduction for Physical Scientists* (Academic Press, Inc., San Diego, London, 1992).
33. D.T. Gillespie, Amer. J. Phys. **61**, 1077 (1993).
34. E.J. Hinch, J. Fluid Mech. **271**, 219 (1994).
35. P.S. Grassia, E.J. Hinch, L.C. Nitsche, J. Fluid Mech. **282**, 373 (1995).
36. D.T. Gillespie, Phys. Rev. E **54**, 2084 (1996).
37. D.T. Gillespie, Amer. J. Phys. **64**, 225 (1996).
38. T.L. Hill, *An Introduction to Statistical Thermodynamics* (Dover Publications, New York, 1986).
39. D.A. McQuarrie, J.D. Simon, *Molecular Thermodynamics* (University Science, Sausalito, Calif., 1999).
40. R. Benzi, A. Sutera, A. Vulpiani, J. Phys. A **14**, L453 (1981).
41. A.R. Bulsara, L. Gammaitoni, Phys. Today **49**, 39 (March 1996).